stratEst: Strategy Estimation in R

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Abstract

stratEst is a software package for the estimation of finite mixture models of discrete choice strategies in the statistical computing environment R. Discrete choice strategies can be customized by the user to fit the environment in which choices are made. The parameters of the strategy estimation model describe the behavior of each strategy and how frequent each strategy is in the population. The estimation function of the package uses the expectation maximization algorithm and the Newton-Raphson method to find the maximum likelihood estimates of the model parameters. The estimation function can also be used to fit a strategy estimation model with individual level covariates to explain the selection of strategies by individuals. The package contains functions for data processing and simulation, strategy generation, parameter tests, model checking, and model selection.

Keywords: discrete choice strategies, finite mixture model, R.

1. Introduction

stratEst is a software package for strategy estimation in the statistical computing environment R (R Development Core Team, 2020). The goal of strategy estimation is to explain discrete choices of a sample of individuals by a finite mixture model of individual choice strategies. Each choice strategy is a complete action plan for all situations that might arise in the choice environment. The parameters of the strategy estimation model describe the behavior of each strategy in the choice environment and how frequent each strategy is in the population.

As a motivating example, consider the following game. Two players play two periods of rock-paper-scissors. In each period of the game, the players simultaneously choose one of three possible actions: rock, paper or scissors. This choice environment creates several situations where a player has to choose between the three alternatives. One situation is the first period of the game where a player has no experience of play. The situation in period two is different. For example, the player might have lost the first period by choosing 'rock'. Or the player might have won the first period by choosing 'paper'. In total, the game can produce $1 + 3 \times 3 = 10$ situations where a player must choose between the three alternatives.

The statement 'I play rock in the first period' describes a choice in this environment. The statement 'I always play rock' describes a strategy for the environment because it defines a choice for every possible situation. The statement 'I randomly choose an action in the first period and subsequently repeat my choice' also describes a strategy for the environment. Each strategy generates a characteristic pattern of choices across the different situations of the choice environment. Comparing the observed choices to the choices predicted by a strategy yields the likelihood that the observed choices were generated by the strategy. This information can be used to (1) refine the behavior of the strategy in order to fit the data better, and (2) estimate the share of individuals in the population which uses the strategy.

Finite mixture models of decision strategies have a long tradition in experimental economics (Stahl and Wilson 1994, 1995). In a seminal contribution, Dal Bó and Fréchette (2011) estimate the maximum likelihood frequencies of a set of candidate strategies to explain participants' choices in a repeated prisoner's dilemma experiment. The study of Dal Bó and Fréchette (2011) established strategy estimation as the standard method to analyze choices in the repeated prisoner's dilemma (e.g. Aoyagi, Bhaskar, and Frechette 2019; Arechar, Dreber, Fudenberg, and Rand 2017; Camera, Casari, and Bigoni 2012; Embrey, Frechette, and Yuksel 2017; Fudenberg, Rand, and Dreber 2012; Frechette and Yuksel 2017). Embrey, Frechette, and Stacchetti (2013) use strategy estimation to explain choices in a repeated partnership game with more than two choice alternatives. Breitmoser (2015) extends the strategy estimation model of Dal Bó and Fréchette (2011) by adding model parameters for the choice probabilities of the strategies. Dvorak and Fehrler (2018) extend the model further by adding individual level covariates to explain the selection of strategies by individuals.

The **stratEst** package provides a general framework for strategy estimation in R. In principle, the package can be used to fit strategy estimation models to any data set with discrete choices. The package overcomes two practical problems of strategy estimation. The first problem is the substantial programming effort needed to perform strategy estimation from scratch. To obtain the maximum likelihood estimates of the model parameters, it is necessary to write code which calculates the likelihood that a given sequence of choices is generated by a certain strategy. If the strategies of the model differ, this requires to write specific code for each strategy. This is a tedious task, especially for complex strategies. The second practical problem of strategy estimation is the difficulty to use existing strategy estimation code for data from other choice environments. The close correspondence between the strategies and the choice environment usually requires to adapt a substantial proportion of the existing code.

The **stratEst** package makes it possible to create, store and adjust choice strategies in a simple format. The strategy generation function of the package represents strategies as deterministic finite state automata. Each automaton has a finite number of internal states. Each state is characterized by a different set of choice probabilities for the choice alternatives. The current state of an automaton changes according to a deterministic rule after receiving some input from the environment. In the automaton representation of a strategy, the probability of each choice alternative depends on the current state of the automaton and not on the choice situation. If the number of choice situations is large, this creates a concise representation of the behavior of the strategy. At the same time, the automaton can mimic complex behavioral patterns by reacting to the input from the environment.

The simplicity of the automaton representation facilitates the programming of strategies considerably. It also makes it easy to adapt existing strategies to other choice environments. Despite the simplicity, even complex strategies can be represented as a deterministic finite state automata. To give an example, consider the strategy known as 'grim trigger' in the literature on the repeated prisoner's dilemma. The strategy suggests cooperation as long as the other player has always cooperated in the past. If the game is repeated for many periods, using the 'grim trigger' strategy requires to correctly remember and adequately react to a long history of past events. Despite the complexity of this task, the strategy can be represented as automaton with only two states. The first state is the start state which prescribes to cooperate. The second state prescribes not to cooperate. It is entered as soon as the partner does not cooperate and never left again.

The estimation function of the package obtains the maximum likelihood parameters of the strategy estimation model based on the expectation maximization algorithm (Dempster, Laird, and Rubin 1977) and the Newton-Raphson method. In principle, this can also be achieved with R packages for cluster and latent class analysis like **Flexmix** (Leisch 2004), **poLCA** (Linzer and Lewis 2011), and **randomLCA** (Beath 2011). A potential drawback of using these packages for strategy estimation is that the model can only contain strategies that are structurally similar to each other. This restriction excludes the possibility to create a reasonable set of candidate strategies for most choice environments. The **stratEst** package overcomes this limitation. The package can be used to fit mixtures of strategies which differ substantially from each other. An important limitation is that it must be possible to represent the strategies of the model as deterministic finite state automata which react to the same inputs from the environment. The package facilitates strategy estimation considerably by providing functions which specifically support strategy programming, data processing and simulation, model selection and checking, and parameter testing.

The package is available from the Comprehensive R Archive Network at http://CRAN.R-project.org/package=stratEst. To speed up the estimation procedure, the package integrates C++ and R with the help of the R packages **Rcpp** (Eddelbuettel and François 2011) and the open source linear algebra library for the C++ language **RppArmadillo** (Sanderson and Curtin 2016). Package development is supported by the packages **devtools** (Wickham, Hester, and Chang 2020b), **testthat** (Wickham 2011), **roxygen2** (Wickham, Danenberg, Csardi, and Eugster 2020a), and **Sweave** (Leisch 2002). Plots are created with **DiagrammeR**, (**??**).

The article is organized as follows. Section 2 formally introduces strategy estimation. Section 3 provides several examples which illustrate the most important functions of the package. Section 4 concludes. Section 5 contains the documentation of the most important functions.

2. Strategy estimation

Strategy estimation is a form of finite mixture modeling (McLachlan and Peel 2005), and similar to cluster analysis (Kaufman and Rousseeuw 1990), and latent class analysis (Lazarsfeld 1950). All methods essentially assign observed entities to unobservable classes. In strategy estimation, the entities are sequences of discrete choices, observed in a specific choice environment, and the unobservable classes are choice strategies.

2.1. Terminology and model definition

Suppose N individuals repeatedly choose between R choice alternatives. Each choice occurs in a certain choice situation $j \in J$. Each situation is characterized by a unique history of observable events. The strategy estimation model assumes that the discrete choices can be explained by a finite mixture of K choice strategies. Each individual i ($i = 1, \dots, N$) uses one of the K strategies. Each strategy k ($k = 1, \dots, K$) assigns a strategy specific state s_k ($s_k = 1, \dots, S_k$) to situation j. The subscript k of the index s_k which indicates that the strategies can have different numbers of states is ignored for better readability. State s determines the probability π_{ksr} that strategy k chooses alternative r in situation j.

Let y^i_{ksr} denote the number of times individual i chooses alternative r in all situations for which the state of strategy k is s. The total number of choices observed in these situations is $n^i_{ks} = \sum_{r=1}^R y^i_{ksr}$. The central modeling assumption of the strategy estimation model is conditional independence (Bandeen-Roche, Miglioretti, Zeger, and Rathouz 1997). Conditional independence implies that the n^i_{ks} choices of individual i are independent conditional on the strategy the individual uses. If individual i uses strategy k, the probability to observe the choice vector $Y^i_{ks} = (y^i_{ks1}, \cdots, y^i_{ksR})$ follows n^i_{ks} independent draws from a multinomial distribution defined by the vector of probabilities $\pi_{ks} = (\pi_{ks1}, \cdots, \pi_{ksR})$ with $\pi_{ksr} \in [0, 1]$ and $\sum_{r=1}^R \pi_{ksr} = 1$.

The basic strategy estimation model

Let p_k denote the share of individuals in the population which follow strategy k defined by the collection of $R \times S_k$ multinomial choice probabilities π_{ksr}. The estimation function of the package returns the estimates p^*_k, π^*_{ksr} that maximize the log likelihood:

$$\ln L = \sum_{i=1}^{N} \ln \left(\sum_{k=1}^{K} p_k \prod_{s=1}^{S_k} \prod_{r=1}^{R} (\pi_{ksr})^{y^i_{ksr}} \right). \tag{1}$$

The parameter constraints are $p_k \in [0, 1]$, $\sum_{k=1}^K p_k = 1$, $\pi_{ksr} \in [0, 1]$ and $\sum_{r=1}^R \pi_{ksr} = 1$. The log likelihood defined in Equation (1) neglects the multinomial coefficients of the likelihood which are constant factors and do not affect the location of the optima.

The strategy estimation model defined by Equation 1 can also contain strategies with pure responses. The response of strategy k in state s is pure, if one alternative is predicted with certainty. In this case, the choice probabilities of strategy k in state s are the result of pure choice probabilities $\xi_{ksr} \in \{0, 1\}$ confounded by trembling hand errors (Selten 1975). Let $\gamma_{ks} \in [0, 1]$ denote the tremble probability of strategy k in state s. The choice probabilities of strategy k in state s are:

$$\pi_{ksr} = \xi_{ksr}(1 - \gamma_{ks}) + (1 - \xi_{ksr})\frac{\gamma_{ks}}{R - 1}. \tag{2}$$

Equation 2 implies that the tremble uniformly implements one of the choices not predicted by the strategy. The tremble rules out that a single choice which is not predicted by the strategy results in a likelihood of zero that the individual uses the strategy. If the model contains strategies with pure responses, the estimation function of the package additionally returns the parameter estimates ξ^*_{ksr} and γ^*_{ks} that maximize the log likelihood defined in Equation 1.

The model with covariates

The strategy estimation model with covariates has two parts: a measurement part and a structural part. The measurement part contains the choice parameters of the strategies and is the same as in the model without covariates. The structural part of the model explains the prior probability p_{ik} that individual i uses strategy k as a function of individual level covariates. The structural part of the model with covariates is the same as in latent class regression (Dayton and Macready 1988; Bandeen-Roche *et al.* 1997).

The structural part uses the first strategy as the benchmark. The log odds of using strategy k compared to the first strategy are modeled by the multinomial logit link function (Agresti 2003). Let x_i denote a row vector that contains the covariates of individual i, then:

$$\ln(p_{ik}/p_{i1}) = x_i \beta_k \;\forall\; k$$

where p_{ik} is the prior probability that individual i uses strategy k and β_k is a column vector of C coefficients. The K equations above yield:

$$p_{ik} = \frac{e^{x_i \beta_k}}{\sum_{k=1}^{K} e^{x_i \beta_k}}$$

The estimation function of the package maximizes the log likelihood:

$$\ln L = \sum_{i=1}^{N} \ln \left(\sum_{k=1}^{K} p_{ik} \prod_{s=1}^{S_k} \prod_{r=1}^{R} (\pi_{ksr})^{y_{ksr}^{i}} \right) \tag{3}$$

The parameters of the model defined in Equation 3 are the parameters of the basic strategy estimation model, except for the strategy shares p_k. The strategy shares are replaced by the vectors β_k which contain the coefficients of the covariates.

The measurement part of the model with covariates assumes local conditional independence. The structural part of the model with covariates assumes non-differential measurement (Bandeen-Roche *et al.* 1997). Non-differential measurement suggests that the choices of all individuals that use the same strategy are not associated with the covariates of these individuals.

When fitting a model with covariates, the parameters in the structural part and the measurement part are estimated simultaneously. This presents an advantage over a two-step estimation. In the two-step estimation, the strategy estimation model is estimated without covariates first and individuals are assigned to strategies on the basis of the posterior probability to use each strategy. In the second step, the classification of individuals is used as the dependent variable in a multinomial model with the individual level covariates as independent variables. It can be shown that the two-step approach suffers from downward biased regression coefficients for the effects of covariates if the classification of individuals is noisy (Bolck, Croon, and Hagenaars 2004).

2.2. Parameter estimation

The estimation function of the package is `stratEst.model()`. The function obtains the maximum likelihood estimates of the model parameters. The model are the strategy shares, the (pure) choice probabilities, the tremble probabilities, and, if the model has covariates, the coefficients of the covariates. The estimated model parameters are returned by the estimation function as objects `shares.par`, `probs.par`, `trembles.par`, and `coefficients.par`.

The estimation function uses the expectation maximization algorithm (EM, Dempster *et al.* 1977) and the Newton-Raphson method to obtain the maximum likelihood estimates of the model parameters. The expectation maximization algorithm exploits the fact that the maximum likelihood estimates of the strategy parameters could be inferred if the assignments of individuals to strategies were known.

The optimization procedure randomly initializes parameters subject to the parameter constraints. After initialization the EM algorithm iterates between two steps until convergence. In the *expectation step* of each iteration, the posterior probability that individual i uses strategy k is updated based on the current values of the model parameters. For the model without covariates, the posterior probability that individual i uses strategy k is a function of the prior probability p_k and the likelihood of the choices given the strategy parameters π_{ksr}.

$$\theta_{ik} = \frac{p_k \prod_{s=1}^{S_k} \prod_{r=1}^{R} (\pi_{ksr})^{y^i_{ksr}}}{\sum_{k=1}^{K} p_k \prod_{s=1}^{S_k} \prod_{r=1}^{R} (\pi_{ksr})^{y^i_{ksr}}} \tag{4}$$

For the model with covariates the prior probability is replaced by the probability p_{ik} which is a function of the covariates.

In the *maximization step* of each iteration, the model parameters are updated to the values that maximize (1) or (3) conditional on the calculated posterior probability assignments of individuals to strategies.

After all model parameters have been updated, the log likelihood of the updated model is determined based on (1) or (3) and compared to the log likelihood calculated in the previous iteration. The algorithm continues with the next iteration as long as the increase in the log likelihood exceeds a certain threshold.

To avoid that local optima are returned by the estimation function, the optimization procedure performs a series of short 'inner' runs of the EM algorithm from different starting points. The best solution obtained in the inner runs is used as the starting point of an 'outer' run of EM. The estimation function of the package returns the best solution obtained in the outer runs. Biernacki, Celeux, and Govaert (2003) show that this method can be used to efficiently locate the maximum likelihood parameters of mixture models. In the maximization step of each iteration, the model parameters are updated according to the following rules.

Strategy shares

In the model without covariates, the population shares p_k are updated to the expected values of the posterior probability assignments of individuals to strategies. The optimization of strategy shares p_k, with respect to a sum-to-one constraint is performed based on the Lagrange multiplier function

$$\Lambda(p_k, \lambda) = \ln L + \lambda \left(\sum_{k=1}^{K} p_k - 1 \right).$$

Setting the partial derivatives $\partial\Lambda/\partial p_k$ and $\partial\Lambda/\partial\lambda$ to zero and solving for p_k and λ yields the conditions

$$p_k = -\sum_{i=1}^{N} \frac{\theta_{ik}}{\lambda} \quad \text{and} \quad \sum_{k=1}^{K} p_k = 1$$

which together yield $\lambda = -N$. Substitution into the first condition yields

$$p_k^{next} = \frac{\sum_{i=1}^{N} \theta_{ik}}{N}. \tag{5}$$

If user defined values are supplied for some strategy shares the remaining strategy shares are scaled by one minus the sum of these values to fulfill the sum-to-one constraint.

Choice probabilities

The choice probabilities π_{ks} are updated based on K weighted data sets. To obtain the weighted data for strategy k, the choices of individual i are considered proportional to the posterior probability θ_{ik} that individual i uses strategy k. Using Lagrange multipliers, the updated choice probabilities π_{ksr} follow from

$$\pi_{ksr^{next}} = \sum_{i=1}^{N} \frac{\theta_{ik} y^i_{ksr}}{\sum_{i=1}^{N} \theta_{ik} n^i_{ks}}. \tag{6}$$

The pure choice probabilities ξ_{ksr} are updated by a transformation of the updated choice probabilities π_{ksr} according to

$$\xi^{next}_{ksr} = \begin{cases} 1 & \text{if } \pi^{next}_{ksr} > \pi^{next}_{ksr'} \ \forall \ r' \neq r \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Equation 7 assigns density of one to the maximum of the updated vector π^{next}_{ks}. This assures that the corresponding tremble parameters γ_{ks} remain as small as possible. If there is more than one parameter with the maximum probability, the first parameter is set to one and the others to zero.

Tremble probabilities

The updated tremble probabilities follow from the substitution of (2) into (6). For the update of the tremble all updated pure choice probabilities affected by the tremble are taken into account.

$$\gamma^{next}_{ks} = \sum_{i=1}^{N} \frac{\theta_{ik} \sum_{r=1}^{R} (y^i_{ksr} - n^i_{ks} \xi^{next}_{ksr}) \left(\frac{R-1}{1-R \cdot \xi^{next}_{ksr}} \right)}{\sum_{i=1}^{N} \theta_{ik} \cdot R \cdot n^i_{ks}}. \tag{8}$$

Regression coefficients

The regression coefficients of the model with covariates are updated based on a Newton-Raphson step (Bandeen-Roche *et al.* 1997). The updated column vector of coefficients β is

$$\beta^{next} = \beta - H_{\beta}^{-1} \nabla_{\beta} \tag{9}$$

where ∇_{β} is the score of the coefficient vector with elements

$$\frac{\partial \ln L}{\partial \beta_{qk}} = \sum_{i=1}^{N} x_{iq} (\theta_{ik} - p_{ik}) \tag{10}$$

in columns and H_{β} is the Hessian of (3) for the coefficients with elements

$$\frac{\partial^2 \ln L}{\partial \beta_{bl} \partial \beta_{ck}} = \sum_{i=1}^{N} x_{ib} x_{ic} (\theta_{il}(\delta_{lk} - \theta_{ik}) - p_{il}(\delta_{lk} - p_{ik})) \tag{11}$$

where $l, k \in \{1, \cdots, K\}$ and $b, c \in \{1, \cdots, C\}$ and $\delta_{lk} = 1$ if $l = k$ and $\delta_{lk} = 0$ otherwise. In order to calculate p_{ik}, for individual i, the row vector x_i which contains the covariates of individual i cannot contain missing values.

Firth (1993) proposes to use the penalized log likelihood function $\ln L^p = \ln L + \frac{1}{2} log(|H_{\beta}|)$ to account for the fact that the maximum likelihood estimates of the coefficients are biased in finite samples. In some instances the maximum likelihood coefficients for the sample can be infinite.

The Firth correction produces reasonable estimates and standard errors in such situations by penalizing the likelihood function with the Jeffreys' invariant prior. Using the penalized likelihood $L^p = L|H_\beta|^{\frac{1}{2}}$ effectively shrinks coefficients towards zero which guarantees that maximum likelihood estimates do exist.

Taking the derivative of the penalized log likelihood with respect to the coefficients yields the penalized score vector $\nabla^p_{\beta_{ah}}$. Bull, Mak, and Greenwood (2002) show that the small sample bias can effectively be reduced by using the penalized score $\nabla^p_{\beta_{ah}}$ to update the coefficients of the multinomial logistic regression model. The penalized score vector of the regression coefficients in the model with covariates is:

$$\nabla^p_{\beta_{ah}} = \frac{\partial \ln L^p}{\partial \beta_{ah}} = \sum_{i=1}^{N} (x_{ia}(\theta_{ih} - p_{ih})) + \frac{1}{2} \operatorname{tr}\left(H_\beta^{-1} \frac{\partial H_\beta}{\partial \beta_{ah}}\right) \tag{12}$$

where $h \in \{1, \cdots, K\}$ and $a \in \{1, \cdots, C\}$ and $\operatorname{tr}(\cdot)$ is the trace of the matrix. The derivative of the element in row l with covariate b and column k with covariate c of the hessian matrix H_β with respect to β_{ah} is:

$$\begin{aligned}\frac{\partial H_\beta}{\partial \beta_{ah}} = \sum_{i=1}^{N} x_{ia}x_{ib}x_{ic}(&\theta_{il}(\delta_{lk} - \theta_{ik})(\delta_{lh} - \theta_{ih}) - \theta_{il}\theta_{ik}(\delta_{kh} - \theta_{ih}) + \\ &p_{il}(\delta_{lk} - p_{ik})(\delta_{lh} - p_{ih}) - p_{il}p_{ik}(\delta_{kh} - p_{ih}))\end{aligned}$$

where $h, l, k \in \{1, \cdots, K\}$ and $a, b, c \in \{1, \cdots, C\}$ and $\delta_{ab} = 1$ if $a = b$ and 0 otherwise.

2.3. Standard errors

The estimation function returns the standard errors of the model parameters as the objects `shares.se`, `probs.se`, `trembles.se`, and `coefficients.se`. The standard errors of the model parameters are estimated based on the empirical observed information matrix (Meilijson 1989)(see Linzer and Lewis 2011, for an earlier application of the method). The empirical observed information matrix is

$$I_e(Y, \hat{\Psi}) = \sum_{i=1}^{N} s(Y_i, \hat{\Psi}) s^T(Y_i, \hat{\Psi}), \tag{13}$$

where $s(Y_i, \hat{\Psi})$ is the score contribution of individual i with respect to parameter vector Ψ, evaluated at the maximum likelihood estimate $\hat{\Psi}$. The reported standard errors are the square roots of the main diagonal of the inverse of $I_e(Y, \hat{\Psi})$.

To calculate the standard error of the parameter η_r with $\sum_{r=1}^{R} \eta_r = 1$, the score function $s(Y_i, \hat{\eta_r})$ transformed into log ratios $\mu_r = ln(\eta_r/\eta_1)$ and the variance-covariance matrix $\mathrm{VAR}(\eta)$ is calculated based on (13). The variance-covariance matrix $\mathrm{VAR}(\mu)$ of the parameters is approximated using the delta method

$$\mathrm{VAR}(f(\hat{\mu})) = f'(\mu) I_e(Y, \hat{\mu})^{-1} f'(\mu)^T, \tag{14}$$

where $f'(\mu)$ is the Jacobian of the function $f(\mu_r) = \eta_r = e^{\mu_r} / \sum_{r=1}^{R} \mu_r$ which converts the values back to the original units.

The following score contributions are used to calculate the empirical observed information matrix defined in (13).

Strategy shares

The shares are transformed into log ratios as $p_k^* = ln(p_k/p_1)$ and the score contribution $\partial \text{ln}L/\partial p_k^*$ of individual i is

$$s(Y_i, p_k^*) = \theta_{ik} - p_k. \tag{15}$$

Let $f(p_k^*) = p_k = e^{p_k^*}/\sum_{l=1}^{K} e^{p_l^*}$ denote the inverse of the transformation, then the Jacobian $f'(p^*)$ has elements

$$\frac{\partial f(p_k^*)}{\partial p_l^*} = \begin{cases} -p_k p_l & \text{if } l \neq k \\ p_k(1-p_l) & \text{if } l = k \end{cases} \tag{16}$$

and the variance-covariance matrix of the shares is estimated by (14) using the inverse of (13) based on the score contributions of the shares defined in (15).

Choice probabilities

If π_{ksr} are mixed parameters standard errors are calculated based on the transformation $\pi_{ksr}^* = ln(\pi_{ksr}/\pi_{ks1})$ and the score contribution $\partial \text{ln}L/\partial \pi_{ksr}^*$ of individual i is

$$s(Y_i, \pi_{ksr}^*) = \theta_{ik} \left(y_{ksr}^i - n_{ks}^i \pi_{ksr} \right). \tag{17}$$

Let $g(\pi_{ksr}^*) = \pi_{ksr} = e^{\pi_{ksr}^*}/\sum_{r=1}^{R} \pi_{ksr}^*$ denote the inverse of the transformation, then the Jacobian $g'(\pi^*)$ has elements

$$\frac{\partial g(\pi_{ksr}^*)}{\partial \pi_{ltq}^*} = \begin{cases} -\pi_{ksr}\pi_{ltq} & \text{if } k = l \text{ and } s = t \text{ and } r \neq q \\ \pi_{ksr}(1-\pi_{ltq}) & \text{if } k = l \text{ and } s = t \text{ and } r = q \\ 0 & \textit{otherwise} \end{cases} \tag{18}$$

and the variance-covariance matrix of the choice probabilities is estimated by (14) using the inverse of (13) based on the score contributions defined in (17).

Tremble probabilities

For strategies with pure responses, the score contribution $\partial \text{ln}L/\partial \gamma_{ks}$ of individual i is

$$s(Y_i, \gamma_{ks}^*) = \theta_{ik} \sum_{r=1}^{R} \frac{y_{ksr}^i}{\pi_{ksr}} \left(\frac{1 - \xi_{ksr}}{R-1} - \xi_{ksr} \right) \tag{19}$$

the reported estimates of the variance-covariance of the tremble probabilities is the inverse of (13) using the score contributions outlined in (19).

Regression coefficients

The reported estimates of the variance-covariance is the inverse of (13) using the score of the regression coefficients outlined in (10) or (12) if the Firth penalty is used.

Bootstrapped standard errors

Standard errors of the model parameters can also be obtained by a parametric bootstrap (Efron and Tibshirani 1993). In each bootstrap sample m ($m = 1, \dots, M$), parameter estimates are obtained based on the observations of N individuals sampled with replacement. Estimates for the choice probabilities π_{ksr} and γ_{ks} for strategy k are obtained by fixing the parameters of all other strategies of the model at the original maximum likelihood estimates. This maintains the original structure of the estimated model across the M bootstrap samples.

For the model with covariates, the Firth penalty is used to obtain the estimates of the regression coefficients for each sample. The reason is that it is likely that some bootstrap samples suffer from quasi complete separation. In a sample with quasi complete separation, the maximum likelihood estimates of the regression coefficients do not exist, and the returned estimates can be infinite. The infinite values bias the estimated standard errors of the regression coefficients. The penalized estimation prevents that infinite parameter values are obtained in samples with quasi complete separation.

2.4. Model fit

The model checking function of the package is `stratEst.check()`. The function returns the log likelihood of the model, the number of free model parameters, and the values of three information criteria. The function can also be used to assess the global and local model fit based on the Pearson χ^2 goodness of fit statistic.

Information criteria

Three different penalized-likelihood criteria can be used to assess the global model fit. The criteria are the Akaike Information Criterion (AIC, Akaike 1973), the Bayesian Information Criterion (BIC, Schwarz 1978), and the Integrated Classification Likelihood (ICL, Biernacki, Celeux, and Govaert 2000). The formulas for the three model selection criteria are:

$$\text{AIC} = -2\ln L + 2df$$

$$\text{BIC} = -2\ln L + log(N_{obs})df$$

$$\text{ICL} = \text{BIC} + 2\sum_{i=1}^{N}\sum_{k=1}^{K}\theta_{ik}log(\theta_{ik})$$

In all three formulas, *df* represents the number of free parameters of the model. The number of free parameters is returned by the estimation function as the object `free.par`. The three information criteria differ in the size of the penalty for model complexity. AIC penalizes the log likelihood with two times the number estimated parameters. BIC penalizes the log likelihood with the number of estimated parameters times the natural logarithm of the number of observations (`num.obs`). ICL uses the BIC penalty plus an extra penalty term for the entropy of the posterior probability assignments of individuals to strategies.

χ^2 test of global fit

The Pearson χ^2 goodness of fit test can be used to assess the global model fit of the strategy estimation model (see van Kollenburg, Mulder, and Vermunt 2015, for an application to latent class models). The Pearson χ^2 goodness of fit test statistic is:

$$\chi^2 = \sum_{k=1}^{K} \sum_{s=1}^{S_k} \sum_{r=1}^{R} \frac{(o_{ksr} - e_{ksr})^2}{e_{ksr}}, \tag{20}$$

where $o_{ksr} = \sum_{i=1}^{N} \theta_{ik} y^i_{ksr}$ and $e_{ksr} = \sum_{i=1}^{N} p_k \pi_{ksr} n^i_{ks}$ represent the observed and the expected number of choices of alternative r by strategy k in state s. As the assignments of individuals to strategies are unknown, the statistic is calculated using the posterior probability assignment θ_{ik} of individual i to strategy k.

The distribution of the test statistic is estimated by a parametric bootstrap (Efron and Tibshirani 1993). The bootstrap procedure simulates M samples of data for the fitted model. In each sample, individuals are randomly assigned to the strategies with probabilities equal to the estimated shares. Choices are simulated conditional on the input observed by each individual and the fitted choice parameters of the strategy the individual uses. The distribution of the test statistic is approximated by calculating the statistic defined in (20) in each of the M samples.

χ^2 test of local fit

The local fit of each strategy is assessed by assigning individuals to strategies based on the maximum values of the posterior probability assignments (Bandeen-Roche *et al.* 1997). Let N_k denote the set of all individuals with a posterior probability maximum for strategy k. The Pearson χ^2 statistic for strategy k is:

$$\chi^2_k = \sum_{i \in N_k} \sum_{s=1}^{S_k} \sum_{r=1}^{R} \frac{(o_{ksr} - e_{ksr})^2}{e_{ksr}} \tag{21}$$

with $o_{ksr} = y^i_{ksr}$ and $e_{ksr} = \pi_{ksr} n^i_{ks}$ as the observed and the expected number of choices of alternative r by strategy k in state s.

The distributions of the K local fit statistics are estimated by a parametric bootstrap. The bootstrap simulates M samples of data for the fitted model. The distribution of the test statistic is approximated by calculating the statistic defined in (20) in each of the M samples.

2.5. Model selection

The number of free model parameters equals $(K-1)+(R-1)\cdot\sum_{k=1}^{K} S_k$ for the model without covariates and $C(K-1)+(R-1)\cdot\sum_{k=1}^{K} S_k$ for the model with covariates. The number of free model parameters changes if the models contains strategies with pure responses. Four different methods can be used to reduce the number of free model parameters.

Parameter fixation

The first method is to fix model parameters at user defined values. The fixation of model parameters can often be justified on the basis of theory. Parameters of all classes of model parameters can be fixed. It is possible to fix the strategy shares, the (pure) choice probabilities, the tremble probabilities, and the coefficients of a model.

It is generally also possible to fix only a subset of parameters of the same class (e.g. two out of four strategy shares). The coefficients of the model with covariates mark an exception.

For this parameter class, either all or no parameter can be fixed. Fixed parameters are not estimated and reduce the number of free model parameters accordingly.

The fixation of parameters which are subject to a sum-to-one constraint affects all other parameters affected by the constraint. If parameters are fixed at a certain value, the remaining parameters are updated and subsequently scaled by the one minus the sum of fixed parameters.

Parameter restrictions

The argument `r.probs` of the estimation function `stratEst.model()` can be used to restrict the number of estimated choice probabilities π. Four options can be used which are `"strategies"`, `"states"`, `"global"`, and `"no"`. The default is `"no"` which implements no restriction.

The option `"strategies"` estimates one parameter vector $\pi_k = (\pi_{k1}, \dots, \pi_{kR})$ for each of the K strategies. The vector π_k determines the probability of choices in all states $s \in \{1, \dots, S_k\}$ of strategy k. The option `"states"` estimates one parameter vector $\pi_s = (\pi_{s1}, \dots, \pi_{sR})$ for each state $s \in \{1, \dots, max(S_k)\}$. The vector π_s determines the probability of choices in state s for all strategies. The option `"global"` estimates a single parameter vector $\pi = (\pi_1, \dots, \pi_R)$ that determines the probability of choices in all states of each strategy.

For strategies with pure choice probabilities, the argument `r.trembles` can be used in a similar fashion. The four options of for the argument `r.trembles` are `"strategies"`, `"states"`, `"global"`, and `"no"`. The default is `"global"` which estimates one tremble probability which applies across all states of all strategies.

The option `"strategies"` estimates one tremble probability γ_k per strategy. The tremble probability γ_k determines the probability of a tremble in all states $s \in \{1, \dots, S_k\}$ of strategy k. The option `"states"` estimates one tremble probability γ_s per state. The tremble probability γ_s determines the probability of a tremble in state s for all strategies. The option `"global"` estimates a single tremble probability γ that determines the probability of a tremble in all states of each strategy. The option `"no"` estimates a tremble probability γ_{ks} for each state of each strategy. The tremble probability γ_{ks} determines the probability of a tremble in state s of strategy k.

If restrictions to the strategy parameters apply, the maximization step in the parameter estimation needs to be adapted accordingly. Let Z_{kt} denote the set of all states s of strategy k where the corresponding strategy parameters are restricted to have the same underlying parameter vector ζ_t, with t ($t = 1, \dots, T$) being the index of the restrictions. The individual score contributions to ζ_t take all parameters affected by restriction t into account, i.e.

$$\pi_{tr}^{next} = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{s \in Z_{kt}} \frac{\theta_{ik} y_{ksr}^{i}}{\sum_{i=1}^{N} \sum_{s \in Z_{kt}} \theta_{ik} n_{ks}^{i}} \tag{22}$$

if ζ_t is a vector of choice probabilities. The tremble probabilities ζ_t are updated according to

$$\gamma_{t}^{next} = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{s \in Z_{kt}} \frac{\theta_{ik} \sum_{r=1}^{R} (y_{ksr}^{i} - n_{ks}^{i} \xi_{ksr}^{next}) \left(\frac{R-1}{1-R \cdot \xi_{ksr}^{next}} \right)}{\sum_{i=1}^{N} \sum_{s \in Z_{kt}} \theta_{ik} \cdot R \cdot n_{ks}^{i}}. \tag{23}$$

The scores of the parameters change accordingly. The score contribution of individual i is the sum over all states $s \in Z_{kt}$ in which parameters are affected by restriction t. The contribution of individual i to the score $\partial \ln L / \partial \pi^*_{tr}$ of the restricted choice probability π^*_{tr} is

$$s(Y_i, \pi^*_{tr}) = \sum_{k=1}^{K} \theta_{ik} \sum_{s \in Z_{kt}} \left(y^i_{ksr} - n^i_{ks} \pi_{ksr} \right). \tag{24}$$

and the Jacobian $g'(\pi^*)$ has elements

$$\frac{\partial g(\pi^*_{tr})}{\partial \pi^*_{uq}} = \begin{cases} -\pi_{tr}\pi_{uq} & \text{if } t = u \text{ and } r \neq q \\ \pi_{tr}(1 - \pi_{uq}) & \text{if } t = u \text{ and } r = q \\ 0 & \textit{otherwise.} \end{cases} \tag{25}$$

The contribution of individual i to the score $\partial \ln L / \partial \gamma_t$ of the restricted tremble probability γ_t of individual i is

$$s(Y_i, \gamma_t) = \sum_{k=1}^{K} \sum_{s \in Z_{kt}} \theta_{ik} \sum_{r=1}^{R} \frac{y^i_{ksr}}{\xi_{ksr}} \left(\frac{1 - \pi_{ksr}}{R - 1} - \pi_{ksr} \right) \tag{26}$$

Parameter selection

The number of choice parameters π and γ can be selected with the argument `select` of the estimation function `stratEst.model()`. The options `"probs"` and `"trembles"` select the number of choice parameters π, and γ respectively. The selection is performed based on one of the three information criteria. The argument which identifies the information criterion is `crit`. Options are `"aic"`, `"bic"` or `"icl"`.

The arguments `r.probs` and `r.trembles` control which combinations of parameter vectors can be reduced to a single parameter vector. The option `"strategies"` defines that the parameter vectors within each strategy are selected. The option `"states"` defines that the parameter vectors within each state across strategies are selected. The option `"global"` defines that all parameter vectors are selected.

The selection procedure starts by estimating the unrestricted model. For every feasible pairwise combination of parameters vectors of the same parameter class, a model is estimated where the two parameter vectors are reduced to a single parameter vector. The lowest value of the information criterion of these models is compared to the value of the information criterion of the unrestricted model. If the model with the reduced number of parameters has a better fit according to the information criterion, it is the new best model. The procedure continues as long as the fusion of any feasible combination of two parameters vectors improves the fit of the model.

Strategy selection

The number of strategies K is selected with the option `select = "strategies"`. The selection is performed based on one of the three information criteria. The argument that identifies the information criterion is `crit`. Options are `"aic"`, `"bic"` or `"icl"`.

The selection procedure starts by estimating the complete model with K strategies. Next, the K nested models with $K - 1$ strategies are estimated. The K nested models are obtained by excluding one strategy from the set of candidate strategies. The best value of the information criterion of the K nested models is compared to the value of the complete model with K strategies. If the value of the nested model is lower, the nested model is the new best model, and new number of strategies is $K - 1$. The selection procedure is repeated as long as the exclusion of one strategy improves the value of the information criterion.

3. Using stratEst

The following examples illustrate the most important functions of the **stratEst** package. The functions are `stratEst.strategy()` for strategy generation, `stratEst.data()` and `stratEst.simulate()` for data processing and simulation, `stratEst.model()` for model estimation, `stratEst.check()` for model checking, and `stratEst.test()` for parameter testing.

3.1. Rock-paper-scissors: An introductory example

This example illustrates the core features of the package on the basis of the game rock-paper-scissors. In each period of this game, two players simultaneously choose one of three possible actions: rock, paper or scissors. The winner of the period is determined by the following rule: rock crushes scissors, scissors cut paper, and paper covers rock. If both players choose the same action, this results in a tie.

This example shows how to fit different strategy estimation models to the data of a rock-paper-scissors experiment conducted by Wang, Xu, and Zhou (2014). The data frame `WXZ2014` contains the data of the experiment. In the experiment, 72 university students play 300 periods of rock-paper-scissors in groups of six. In each period, each student is randomly matched with another student from the same group. 35.7 percent of all choices in the data are rock (`r`), 32.2 percent are paper (`p`), and 32.1 percent are scissors (`s`). The distribution of choices is fairly in line with the Nash equilibrium of rock-paper-scissors. The Nash equilibrium suggest that every player uses the same strategy. This strategy plays each of the three actions with probability one-third.

Models with mixed strategies

The function `stratEst.strategy()` is the strategy generation function of the package. The strategy function returns a data frame object of class `stratEst.strategy`. The data frame object represents a strategy as deterministic finite state automaton. Each row of the data frame is one state of the automaton. Each state assigns a probability to each of the three choice alternatives: rock (`r`), paper (`p`), and scissors (`s`).

The following code creates a mixed strategy for the rock-paper-scissors game. The mixed strategy plays each of the three alternatives `r`, `p`, and `s` with some unknown probability.

```
R> library(stratEst)
R> rps = c("r", "p", "s")
R> mixed = stratEst.strategy(choices = rps)
R> print(mixed)
```

```
  prob.r prob.p prob.s
1     NA     NA     NA
```

The argument `choices` defines the names of the choice alternatives. Printing the object `mixed` to the console, shows that the strategy `mixed` is represented as an automaton with one state. The probabilities of the three choices `r`, `p`, and `s` in this state are `NA`. If a model with the strategy `mixed` is fitted, the choice probabilities are parameters of the model. Whenever a model parameter is `NA`, this indicates to the estimation function that the parameter should be estimated from the data.

The estimation function `stratEst()` can be used to fit a model with the strategy `mixed` to the rock-paper-scissors data.

```
R> model.one.mixed <- stratEst.model(data = WXZ2014,
+                                   strategies = list("mixed" = mixed))
```

The estimation function `stratEst()` checks the supplied arguments, estimates the model and returns a list object of class `stratEst.model`. The elements of this list can be accessed with the syntax `model$object` where object can be replaced by any object name in `names(model)`. The generic function `summary()` can be used to print a summary of the estimated model to the console.

```
R> summary(model.one.mixed)

model.one.mixed
---------------
number of individuals: 72
number of observations: 21600
log likelihood: -23704.04
free model parameters: 2
convergence: yes

shares
------
      mixed
share     1

strategies
----------
      prob.r prob.p prob.s
mixed   0.36   0.32   0.32
```

Since the model has only one strategy, the estimated share of this strategy is one. The maximum likelihood estimates of the choice probabilities of the mixed strategy reflect the overall distribution of choices in the data.

The following command fits a mixture model of two mixed strategies.

```
R> set.seed(0)
R> model.two.mixed <- stratEst.model(data = WXZ2014,
+                                   strategies = list("mixed.1" = mixed,
+                                                     "mixed.2" = mixed))
R> print(model.two.mixed$shares)
```

```
         mixed.1   mixed.2
share 0.06944161 0.9305584
```

```
R> print(model.two.mixed$strategies)
```

```
$mixed.1
     prob.r    prob.p    prob.s
1 0.2720114 0.2026563 0.5253324

$mixed.2
     prob.r    prob.p    prob.s
1 0.3628844 0.3311944 0.3059211
```

The strategy shares can be interpreted as the share of individuals in the population that follows each strategy. According to the fitted model parameters, 93 percent of the individuals in the population follows the strategy `mixed.2`. The choice probabilities of the strategy `mixed.2` are similar to the overall distribution of choices. The remaining 7 percent of the population follows the strategy `mixed.1`. This strategy tends to play scissors and avoids to play paper.

Mixture of two different strategies

The following command creates the Nash equilibrium strategy of rock-paper-scissors.

```
R> nash = stratEst.strategy(choices = rps, prob.choices = c(1/3, 1/3, 1/3))
```

The argument `prob.choice` of the strategy generation function defines the choice probabilities of the strategy `nash`. Therefore, the values supplied to the argument `prob.choice` must sum to one. If the choice probabilities of a strategy are defined, the probabilities are known parameters and do not need to be estimated from the data. If printed out in the console, the strategy `nash` looks like this:

```
R> print(nash)
```

```
  prob.r prob.p prob.s
1  0.333  0.333  0.333
```

The following command fits a mixture of the strategies `mixed` and `nash` to the rock-paper-scissors data.

```
R> model.mixed.nash <- stratEst.model(data = WXZ2014,
+                                     strategies = list("mixed" = mixed,
+                                                       "nash" = nash))
R> print(model.mixed.nash$shares)
```

```
           mixed      nash
share 0.05555565 0.9444444
```

```
R> print(model.mixed.nash$strategies$mixed)

     prob.r    prob.p   prob.s
1 0.3058335 0.1583334 0.535833
```

The returned strategy shares indicate that 94 percent of the population follow the Nash strategy. A minority of 6 percent follows the mixed strategy which tends to play scissors and avoids to play paper.

Complex strategies

The strategies `nash` and `mixed` are simple strategies in the sense that the choice probabilities of these strategies do not change across different situation of the choice environment. Both strategies can be represented as a deterministic finite state automaton with one state. The objects `mixed` and `nash` generated by the function `stratEst.strategy()` have one row which contains the choice probabilities of this state.

More complex strategies can be represented as automata with several states. Each state is represented by one row with a different set of choice probabilities. Transitions between the states are triggered by some input. The input indicates a certain condition of the choice environment or a certain history of past events.

The strategy `imitate` is an example of a more complex strategy. The strategy randomly makes a choice in the first period of the game and subsequently imitates the choice of the previous period. The following code generates the strategy `imitate`:

```
R> last.choice = c(NA, rps)
R> imitate =  stratEst.strategy(choices = rps, inputs = last.choice,
+                                num.states = 4,
+                                prob.choices = c(1/3, 1/3, 1/3, 1, 0, 0,
+                                                 0, 1, 0, 0, 0, 1),
+                                tr.inputs = rep(c(2, 3, 4), 4))
R> rownames(imitate) <- c("mix","rock","paper","scissors")
```

The argument `inputs` of the strategy generation function defines a set of inputs for the strategy. The object supplied to the argument `inputs` is a character vector with one element for each possible choice in the previous period. The value `NA` indicates that the input can be missing. This is the case in the first period when no information about the previous period exists. The argument `num.states` defines the number of states of the automaton. The argument `tr.inputs` defines the deterministic state transitions for all possible inputs and all states. The result is a strategy with four states. Each state is represented by one row of the object `imitate`.

```
R> print(imitate)

         prob.r prob.p prob.s tremble tr(r) tr(p) tr(s)
mix       0.333  0.333  0.333      NA     2     3     4
rock      1.000  0.000  0.000      NA     2     3     4
paper     0.000  1.000  0.000      NA     2     3     4
scissors  0.000  0.000  1.000      NA     2     3     4
```

The object `imitate` has a column named `tr(x)` for each possible input `x`. An exception is the element `NA` that indicates the missing input in the first period. The values supplied to the argument `tr.inputs` appear in row wise order.

The strategy `imitate` also contains a column with the name `tremble`. The values in this column indicate the probability to choose one of the choices not prescribed by the strategy in the current state. A tremble probability is necessary if choice probabilities of zero and one. The purpose of the tremble probability is to avoid that a single deviation from the choice pattern of the strategy results in a likelihood of zero that the individual uses the strategy. The tremble probability can be specified with the argument `trembles`. If the argument is missing, the probability of a tremble is `NA`. This signals to the estimation function that the tremble probabilities need to be estimated from the data.

The strategy `imitate` transitions from one state to the other by the following deterministic rule. In the first period, the strategy observes the input `NA` since there is no information on the previous choice available. By convention, whenever the input is `NA`, the strategy moves to its start state. The start state of the strategy is the state represented by the first row. The strategy makes a choice according to the choice probabilities in the first row.

In period two, the strategy observes the input (either `r`, `p` or `s`) and moves to the next state defined by the value of `tr(input)` in the current state. The values supplied to `tr.inputs` create the desired behavior of the strategy `imitate`. The strategy randomly makes a choice in the first period, and subsequently plays rock after rock, paper after paper, and scissors after scissors.

For complex strategies, the print output can be difficult to understand. A better way to understand complex strategies is to plot them. The command `plot(imitate)` produces a graphical representation of the strategy shown in Figure 3.1.2. This is the automaton representation of the strategy `imitate` rendered as SVG file.

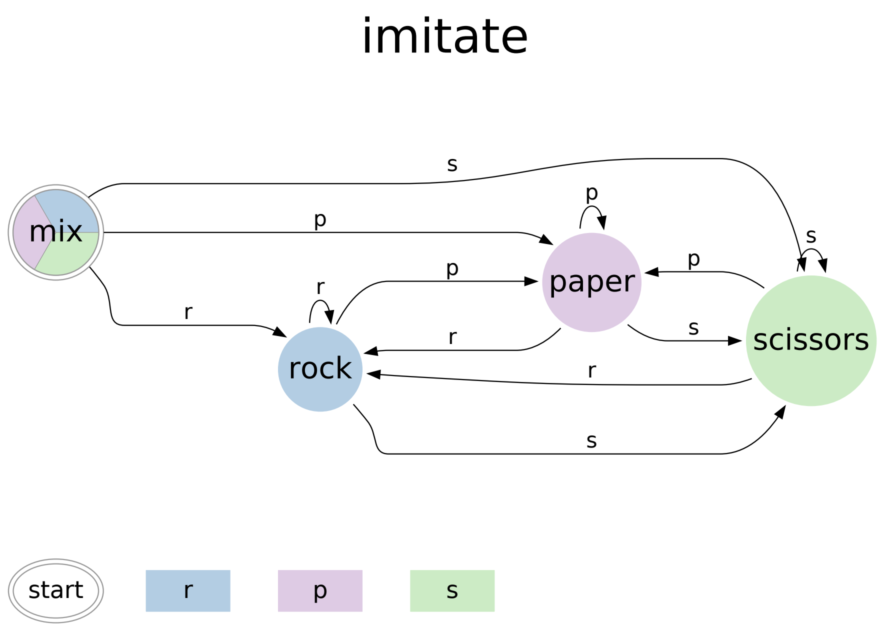
```
R> plot(imitate)
```

In order to fit the strategy `imitate` to the rock-paper-scissors data, the data must contain a variable with the input. The function `stratEst.data()` can be used to generate a data frame with this variable.

```
R> data.WXZ2014 <- stratEst.data(data = WXZ2014, choice = "choice",
+                                 input = c("choice"), input.lag = 1,
+                                 id = "id", game = "game",
+                                 period = "period")
```

The first argument of the function expects a `data.frame` object with variables in columns. The argument `choice` defines the variable in the data which contains the discrete choices. The argument `input` allows selecting one or more variable names which serve as input for the strategies in the estimation. If more than one variable is selected, the function concatenates the values of these variables to a unique factor level. The input of for the strategy `imitate` only needs one variable. This variable is the variable `choice` which contains the choices of the participant. As the input should reflect the value of this variable in the previous period, the argument `input.lag` is set to one. The arguments `id`, `game`, and `period` uniquely identify the participant, and the period within the game.

Figure 1: Graphical representation of a strategy



The function `stratEst.data()` returns a data frame object of class `stratEst.data`. The command `print(data.WXZ2014)` prints the object.

The following command estimates a mixture model with the strategies `nash` and `imitate`.

```
R> model.nash.imitate <- stratEst.model(data = data.WXZ2014,
+                                   strategies = list("nash" = nash,
+                                                     "imitate" = imitate))
R> print(model.nash.imitate$shares)

           nash   imitate
share 0.5843183 0.4156817

R> print(model.nash.imitate$strategies)

$nash
     prob.r    prob.p    prob.s tr(p) tr(r) tr(s) tremble
1 0.3333333 0.3333333 0.3333333     1     1     1      NA

$imitate
         prob.r prob.p prob.s tremble tr(r) tr(p) tr(s)
mix       0.333  0.333  0.333      NA     2     3     4
rock      1.000  0.000  0.000   0.391     2     3     4
paper     0.000  1.000  0.000   0.391     2     3     4
scissors  0.000  0.000  1.000   0.391     2     3     4
```

The estimated shares suggest that each strategy is used by approximately half of the participants in the experiment. The fitted tremble parameter of the strategy `imitate` indicates that a different choice than the one predicted by the strategy is chosen in 39 percent of all observations. In these observations, the participants randomly pick one of the choice alternatives not predicted by the strategy.

Model fit

The function `stratEst.check()` can be used to check the global and local fit of a model. The following code summarizes the log likelihood, the number of free model parameters, and the values of three information criteria of the models estimated so far.

```
R> models <- list(model.one.mixed, model.two.mixed, model.mixed.nash,
+                 model.nash.imitate)
R> model.fit <- do.call(rbind, unlist(lapply(models, stratEst.check),
+                                   recursive = F))
R> rownames(model.fit) <- c("model.one.mixed", "model.two.mixed",
+                           "model.mixed.nash", "model.nash.imitate")
R> print(model.fit)
```

```
                    loglike free.par      aic      bic      icl
model.one.mixed    -23704.04        2 47412.09 47416.64 47416.64
model.two.mixed    -23574.93        5 47159.86 47171.24 47171.25
model.mixed.nash   -23613.30        3 47232.60 47239.43 47239.43
model.nash.imitate -22358.43        2 44720.87 44725.42 44728.34
```

The best global fit is obtained by the mixture of `nash` and `imitate`. The log likelihood of this model is larger than the log likelihood of the other models. The mixture model of `nash` and `imitate` also yields the best (smallest) values of the three information criteria: the Akaike information criterion (`aic`), the Bayesian information criterion (`bic`), and Integrated classification likelihood (`icl`). The information criteria take the number of free model parameters into account for the evaluation of the global model fit.

Model selection

The argument `select` of the estimation function can be used for model selection. To give an example, the option `select = "strategies"` selects the best of all models which can be constructed by combination of six rock-paper-scissors strategies. The list `strategies.RPS` contains the six rock-paper-scissors strategies. The strategies are the strategies `mixed`, `nash`, `imitate`, `rock`, `paper`, and `scissors`. The strategies `rock`, `paper`, and `scissors` always make the same choice if no tremble occurs. The argument `crit` of the estimation function defines the information criterion used for the selection. The model selection procedure starts by estimating the model that contains all six strategies and subsequently drops strategies as long as this improves the value of the information criterion.

```
R> selected.model <- stratEst.model(data = data.WXZ2014,
+                                   strategies = strategies.RPS,
+                                   select = "strategies", crit = "aic")
R> print(selected.model$shares)
```

```
            rock      nash     mixed   imitate
share 0.02785163 0.4069174 0.1628454 0.4023855
```

The best model identified by the selection procedure contains the four strategies `rock`, `nash`, `mixed`, and `imitate`. A closer look at the strategies `rock` and `imitate` reveals that these two strategies have the same tremble probability. The reason is that, by default, the estimation function estimates one tremble parameter for all strategies and all states.

To argument `r.trembles` of the estimation function can be used to relax this restriction. The option `"strategies"` will estimate one tremble parameter per strategy. The option `"state"` one tremble parameter for each state. The option `"no"` estimates tremble parameter for each state of each strategy. Similarly, the argument `r.probs` defines restrictions for the choice probabilities.

```
R> selected.strategies <- strategies.RPS[c("rock","nash","mixed","imitate")]
R> unrestricted.model <- stratEst.model(data = data.WXZ2014,
+                                       strategies = selected.strategies,
+                                       r.trembles = "no")
```

The unrestricted model contains four tremble parameters. One for the strategy `rock`, and three for the strategy `imitate`.

Testing model parameters

The general syntax to access the estimated parameters `x` of a model is `model$x.par`. The estimation function returns several other objects for the estimated parameters `x`. These include the standard errors (`se`), the covariance matrix (`covar`), and the score vector of the parameters (`score`). The returned objects can be accessed with the general syntax `model$x.object`.

The function `stratEst.test()` of the package retrieves the parameter estimates and the standard errors of a fitted model and performs a t test. The degrees of freedom used by the test are the number of individuals minus the number of free model parameters. The degrees of freedom are returned by the estimation function as the object `res.degrees`.

To give an example, the estimates and standard errors of the choice probabilities of the model with one mixed strategy are returned as the object `model.one.mixed$probs.par` and the object `model.one.mixed$probs.se`. The function `stratEst.test()` can be used to show that the estimated choice probabilities differ from the choice probabilities of the Nash strategy.

```
R> test <- stratEst.test(model = model.one.mixed, par = "probs", values = 1/3)
R> print(test)
```

```
            estimate    diff std.error  t.value df p.value
probs.par.1   0.3223 -0.0111    0.0014  -8.0838 70       0
probs.par.2   0.3566  0.0232    0.0013  17.6404 70       0
probs.par.3   0.3212 -0.0122    0.0012 -10.3417 70       0
```

The argument `par` of the function `stratEst.test()` defines the class of model parameters for which tests are performed. The argument `values` defines the values the parameter estimates are compared to.

The function returns a data frame with one row for each tested model parameter. The first column contains the parameter estimates. The second column the differences to the values the parameters are compared to. Columns three to five contain the standard errors of the parameters, the t values and the degrees of freedom of the tests. Columns six contains the p values of the tests which indicate that the choice probabilities differ from one third. It should be noted that the three tests are not independent as the three tested parameters are subject to the same sum-to-one constraint.

Individual-level covariates

The basic strategy estimation model can be augmented by individual level covariates which explain the selection of strategies by individuals. The following commands generate two covariates for the rock-paper-scissors data. The first covariate is an intercept which is one for every individual. The second covariate is a normally distributed random number with a mean of zero and a standard deviation of one. Since the two covariates vary on the level of the individual, observation with the same id need to have the same covariate values.

```
R> set.seed(0)
R> data.covariates <- data.WXZ2014
R> data.covariates$intercept = rep(1, nrow(WXZ2014))
R> id.random <- rnorm(length(unique(WXZ2014$id)))
R> data.covariates$random = id.random[WXZ2014$id]
```

The following command fits a mixture of the strategies `nash` and `imitate` with the covariates `intercept` and `random` to the rock-paper-scissors data.

```
R> model.covariates <- stratEst.model(data = data.covariates,
+                                     strategies = list("nash" = nash,
+                                                       "imitate" = imitate),
+                                     covariates = c("intercept","random"))
R> print(model.covariates$coefficients)

          nash    imitate
intercept    0 -0.3534746
random       0 -0.2531341
```

In a model with covariates, the first strategy in the list of strategies is used as the reference strategy. The coefficients of the reference strategy are always zero. The estimated coefficient of the intercept for the strategy `imitate` is estimated from the data. It is the logarithm of the ratio of the estimated prior probability to use the strategy `imitate` and the estimated prior probability to use the strategy `nash`, if all other covariates are zero.

The coefficient of the intercept can be used to calculate the estimated prior probability to use the strategy `imitate` for an individual with a random covariate of zero. The estimated prior probability to use the strategy `imitate` for this individual is $exp(-0.35)/(1+exp(-0.35)) = 0.41$. Since zero is the expected value of the distribution of the random covariate, this corresponds to the estimated share of the strategy `imitate`.

The coefficient of the covariate `random` can be used to calculate the prior probability for different values of the random covariate. The negative coefficient means that the prior probability to use the strategy `imitate` decreases for larger values of the variable `random`. For instance, the prior probability to use the strategy `imitate` for an individual with a random covariate of one is $exp(-0.35 - 0.25)/(1 + exp(-0.35 - 0.25)) = 0.35$.

3.2. Simulated data

The simulation function of the package is `stratEst.simulate()`. The function can be used to generate data on the basis of a fully specified model. A fully specified model can be obtained by fitting a model to data or by defining each parameter of the model by hand. This example illustrates how the simulation function can be used to validate the parameter estimates and standard errors returned by the estimation function. Consider a model with two strategies for the choices 'left' and 'right':

```
R> set.seed(1)
R> lr <- c("left","right")
R> mixed <- stratEst.strategy(choices = lr, inputs = lr, num.states = 1)
R> pure <- stratEst.strategy(choices = lr, inputs = lr,
+                            prob.choices = c(1, 0, 0, 1),
+                            tr.inputs = c(1, 2, 1, 2))
R> strategies <- list("mixed" = mixed, "pure" = pure)
```

Strategy `mixed` plays 'left' with a mixed probability π drawn from $U(0, 1)$. Strategy `pure` plays 'left' if the input is 'left', and 'right' if the input is 'right' with tremble probability γ drawn from $U(0, 0.25)$.

```
R> pi <- runif(1)
R> gamma <- runif(1)/4
```

The strategy shares depend on the coefficient β drawn from $N(0, 1)$. The value of β defines the share of the mixed strategy p.

```
R> beta <- rnorm(1)
R> p <- 1/sum(1 + exp(beta))
```

The following commands insert the parameters π and γ into the strategies `mixed` and `pure` and create a list of the two strategies.

```
R> p <- 1/sum(1 + exp(beta))
R> sim.shares <- c(p, 1-p)
R> mixed$prob.left <- pi
R> mixed$prob.right <- 1 - pi
R> pure$tremble <- gamma
R> sim.strategies <- list("mixed" = mixed, "pure" = pure)
```

Now, the model is fully specified and can be used to simulate a data set.

```
R> sim.data <- stratEst.simulate(strategies = sim.strategies,
+                                  shares = sim.shares, num.ids = 100,
+                                  num.games = 10, num.periods = rep(5, 10))
```

The simulation function creates a `stratEst.data` object which contains the observations of 100 individuals. Each individual is assigned to one of the strategies with probabilities defined by the object `sim.shares`. Each individual plays ten games with five periods. In each period, the individual observes an input randomly drawn from the set of inputs. The input triggers a state transition of the strategy used by the individual. After the state transition, the individual chooses an alternative with probabilities defined by the current state of the strategy.

The following commands estimate two models. One model without covariates, and one model with an intercept as covariate.

```
R> model <- stratEst.model(data = sim.data, strategies = strategies)
R> sim.data$intercept <- rep(1, nrow(sim.data))
R> model.lcr <- stratEst.model(data = sim.data, strategies = strategies,
+                                covariates = "intercept")
```

The estimated model parameters differ from the true parameters due to sampling error. The function `stratEst.test()` can be used to test if the estimated parameters differ from the true parameters.

```
R> pars <- c(p, 1-p, pi, 1-pi, gamma)
R> test.pars <- stratEst.test(model, values = pars)
R> print(test.pars)

               estimate    diff std.error t.value df p.value
shares.par.1     0.4300 -0.0242    0.0495 -0.4892 97  0.6258
shares.par.2     0.5700  0.0242    0.0495  0.4892 97  0.6258
probs.par.1      0.2716  0.0061    0.0094  0.6535 97  0.5150
probs.par.2      0.7284 -0.0061    0.0094 -0.6535 97  0.5150
trembles.par.1   0.0923 -0.0008    0.0057 -0.1327 97  0.8947
```

The p values of the tests indicate that the tests of the strategy shares and the choice probabilities are not independent. The reason is of course that these parameters are subject to a sum-to-one constraint.

The simulation function generates a variable `strategy` that contains the result of the probabilistic assignment of individuals to strategies. This variable can be used to calculate the maximum likelihood parameters of the sample.

```
R> strategy <- sim.data$strategy
R> choice <- sim.data$choice
R> input <- sim.data$input
R> p.ml <- mean(strategy == "mixed")
R> pi.ml <- mean(choice[strategy == "mixed"] == "left")
R> gamma.ml <- mean( choice[strategy == "pure"] != input[strategy == "pure"])
R> print(round(c(p.ml, 1 - p.ml, pi.ml, 1 - pi.ml, gamma.ml), 4))
```

```
[1] 0.4300 0.5700 0.2716 0.7284 0.0923
```

Comparing the parameter estimates summarized in the output of the function `strategy.test()` to these values shows that the estimation function returned the maximum likelihood parameters of the sample.

Table 1 summarizes the estimation results for 10.000 simulated samples of data. The first three rows of Table 1 depict the results for the parameters of the model without the intercept. The last three rows depict the results for the parameters of the model with the intercept. The columns show the means of the estimated parameters, the difference and absolute difference of the estimated parameters and the maximum likelihood parameters of the sample, and the rejection probability of a t test for the model parameters with an α level of 5 percent.

The first column shows that the means of the estimated parameters are close to the means of the distributions these parameters are sampled from. Columns two and three show that the algorithm usually converges to the maximum likelihood estimates of the sample. Columns four and five show that the rejection rate of t tests of the model parameters is close to the 5 percent α level, for both analytic and bootstrapped standard errors.

				$\mathsf{P}(> \|t\|) < 0.05$	
θ	$\hat{\theta}_m$	$\theta_m - \hat{\theta}_m$	$\|\theta_m - \hat{\theta}_m\|$	analytic	bootstrap
model without covariates					
p	0.4966	6e-05	0.0012	0.0537	0.0551
π	0.5002	2e-06	0.0002	0.0439	0.0558
γ	0.1265	-3e-05	0.0004	0.0497	0.0613
model with covariates					
β	-0.0012	-4e-02	1.6639	0.0442	0.0468
π	0.5002	2e-06	0.0002	0.0439	0.0553
γ	0.1265	-3e-05	0.0004	0.0497	0.0612

Table 1: Estimates and rejection probability for simulated data. Average estimates and rejection probability across 10.000 Monte Carlo samples of simulated data. Each sample contains the choices of 100 individuals in 10 games with 5 periods. θ_m is the maximum likelihood estimate of the parameter in sample m. $\hat{\theta}_m$ is the parameter estimate returned by the estimation function for sample m. Bootstrapped standard errors are based on 100 samples.

3.3. Replication of Dal Bo and Frechette, 2011

This example replicates the strategy estimation results of the seminal strategy estimation by Dal Bó and Fréchette (2011). The study investigates the evolution of cooperation in the indefinitely repeated prisoner's dilemma across six experimental treatments. The six treatments differ in the stage game parameters and the continuation probability δ of the repeated game.

The stage game parameters are depicted in Figure 2. The parameter R is either 32, 40 or 48. For each value of R, two treatments exist with δ of $1/2$ or $3/4$. This results in a 2×3 between subject design with six treatments overall. Dal Bó and Fréchette (2011) fit the same strategy

	C	D
C	R,R	12,50
D	50,12	25,25

Figure 2: Stage game of Dal Bó and Fréchette (2011). The two choices of the stage game are cooperation (C) and defection (D). R varies between experimental treatments and can take the values 32, 40 or 48.

estimation model with six strategies to the data of each treatment. The strategies of this model are: Always Defect (ALLD), Always Cooperate (ALLC), Tit-For-Tat (TFT), Grim-Trigger (GRIM), Win-Stay-Lose-Shift (WSLS), and a trigger strategy with two periods of punishment (T2). The list `strategies.DF2011` contains the six strategies. All six strategies in this list have the same structure. For example, the Tit-For-Tat strategy looks like this:

```
R> print(strategies.DF2011$TFT)
```

```
  prob.d prob.c tremble tr(cc) tr(cd) tr(dc) tr(dd)
1      0      1      NA      1      2      1      2
2      1      0      NA      1      2      1      2
```

The strategy TFT chooses between the alternatives defect (`d`) and cooperate (`c`). State transitions are triggered by four different inputs. The four inputs reflect all possible combinations of actions in the last period. The first letter represents the own action in the previous period and the second letter the action of the other player in the previous period.

The data frame `DF2011` contains the experimental data collected by Dal Bó and Fréchette (2011). The following code creates a `stratEst.data` frame which fits the structure of the strategies in the list `strategies.DF2011`:

```
R> data.DF2011 <- stratEst.data(data = DF2011, choice = "choice",
+                                input = c("choice","other.choice"),
+                                input.lag = 1)
```

The options `input = c("choice", "other.choice")` and `input.lag = 1` create the input variable by concatenating the own and the other player's choice of the previous period.

The command replicates the results of the strategy estimation by Dal Bó and Fréchette (2011):

```
R> model.DF2011 <- stratEst.model(data = data.DF2011,
+                                  strategies = strategies.DF2011,
+                                  sample.id = "treatment" )
```

The option `sample.id = "treatment"` is used to estimate a model with one vector of shares and one tremble parameter for each level of the factor `treatment` in the data. The estimated shares are the strategy shares reported in Table 7 on page 424 of Dal Bó and Fréchette (2011).

```
R> print(round(do.call(rbind, model.DF2011$shares), 2))

                  ALLD ALLC GRIM  TFT WSLS   T2
treatment.D5R32   0.92 0.00 0.00 0.08 0.00 0.00
treatment.D5R40   0.78 0.08 0.04 0.10 0.00 0.00
treatment.D5R48   0.53 0.07 0.00 0.38 0.02 0.00
treatment.D75R32 0.65 0.00 0.00 0.35 0.00 0.00
treatment.D75R40 0.11 0.30 0.27 0.33 0.00 0.00
treatment.D75R48 0.00 0.08 0.12 0.56 0.00 0.24
```

3.4. Replication of Fudenberg, Rand and Dreber, 2012

This example replicates the strategy estimation results of Fudenberg *et al.* (2012). The authors conduct a prisoner's dilemma experiment in which intended choices are implemented with noise. The stage game payoffs are such that cooperation means paying a cost c to give a benefit b to the other player. The authors run four between subjects treatments. The cost c is fixed at 2 points experimental currency in every treatment. The benefit to cost ratio b/c varies across treatments and takes the values 1.5, 2, 2.5, and 4.

Due to the noisy implementation of choices, Fudenberg *et al.* (2012) add several lenient and forgiving strategies to the set of strategies used by Dal Bó and Fréchette (2011). The augmented set of strategies is available as the list `strategies.FRD2012`. The strategies have the same structure as in the previous example. The choices of the strategies are `d` (defect), and `c` (cooperate). The four inputs reflect the four different combinations of the own choice, and the choice of the other player in the previous period.

The data frame `FRD2012` contains the data of the experiment. The variable `choice` contains the choices of the participants. The variables `last.choice` and `last.other` indicate own choice and the choice of the other player in the previous period. These two variables are supplied to the argument `input` of the data generation function:

```
R> data.FRD2012 <- stratEst.data(data = FRD2012, choice = "choice",
+                                  input = c("last.choice", "last.other"))
```

The following code reproduces the strategy shares reported by Fudenberg *et al.* (2012) in Table 3 on page 733 of the paper.

```
R> model.FRD2012 <- stratEst.model(data = data.FRD2012,
+                                    strategies = strategies.FRD2012,
+                                    sample.id = "bc")
R> print(round(do.call(rbind, model.FRD2012$shares), 2))

       ALLC  TFT TF2T TF3T T2FT T2F2T GRIM GRIM2 GRIM3 ALLD DTFT
bc.1.5 0.00 0.19 0.05 0.01 0.06  0.00 0.14  0.06  0.06 0.29 0.14
bc.2   0.03 0.06 0.00 0.03 0.07  0.11 0.07  0.18  0.28 0.17 0.00
bc.2.5 0.00 0.09 0.17 0.05 0.02  0.11 0.11  0.02  0.24 0.14 0.05
bc.4   0.07 0.09 0.18 0.13 0.05  0.09 0.06  0.07  0.10 0.14 0.03
```

For the treatment with $b/c = 4$, the estimation function of the package finds a better solution with a larger log likelihood than the solution reported by Fudenberg *et al.* (2012).

3.5. Replication of Dvorak, Fischbacher and Schmelz, 2020

This example replicates the strategy estimation results of Dvorak, Fischbacher, and Schmelz (2020). The authors study conformity and anticonformity in a binary choice experiment. Participants are matched in groups of three and compete for a monetary reward with the other two group members. One individual of each group is informed about the choices of two other group members before making the own choice. The experimental design makes it possible to predict the preferred alternative of the individual in this choice.

Dvorak *et al.* (2020) find that two-thirds of the participants use a conformist strategy. The conformist strategy generally follows the own preference if the choices of the other group members are in line with the own preference. It frequently deviates from the own preference if the choices of the other group members are not in line with the own preference.

The remaining one-third of the participants follows an anticonformist strategy. The anticonformist strategy generally follows the own preference if the choices of the other group members are not in line with the own preference. It frequently deviates from the own preference if the choices of the other group members are in line with the own preference.

The fitted choice probabilities of the two strategies are:

```
R> print(strategies.DFS2020)

$anticonformist
  prob.follow prob.deviate tr(not in line) tr(in line)
1       0.823        0.177               1           2
2       0.404        0.596               1           2

$conformist
  prob.follow prob.deviate tr(not in line) tr(in line)
1       0.425        0.575               1           2
2       0.860        0.140               1           2
```

The data `DFS2020` contains the experimental data of Dvorak *et al.* (2020). The variable `"choice"` of this data set is a factor with two levels. The two levels indicate if the choice of the participant follows the own preference (`follow`) or deviates from the own preference (`deviate`). The variable `"others.choices"` of this data set is also a factor with two levels. The levels indicate if the choices of the other two group members are in line with the preference of the participant (`in line`) or not in line with the preference of the participant (`not in line`).

The data also contains two variables which are used by Dvorak *et al.* (2020) as individual level covariates for the selection of strategies. The first variable is an intercept, which is one for every observation. The second variable is the score of the participant in a post-experimental conformity questionnaire (Mehrabian and Stefl 1995). The mean conformity score of the participants is -0.078 with a standard deviation of 1.02.

The following command creates a `stratEst.data` object with the variable `others.choices` as input:

```
R> data.DFS2020 <- stratEst.data(data = DFS2020,
+                                input = "others.choices")
```

The following command fits a strategy estimation model with the covariates `intercept` and `conformity.score` to the data:

```
R> model.DFS2020 <- stratEst.model(data = data.DFS2020 ,
+                                  strategies = strategies.DFS2020,
+                                  covariates = c("intercept",
+                                                 "conformity.score"))
```

The estimated coefficients are:

```
R> print(model.DFS2020$coefficients)
```

```
                 anticonformist conformist
intercept                     0  0.8273618
conformity.score              0  0.8697285
```

The first strategy is the reference category of the structural model. The coefficients for the reference category are always zero. The second column contains the estimated coefficients for the conformist strategy. The coefficient of the intercept indicates that the conformist strategy is used more frequently. The estimated coefficient of the intercept can be used to calculate the estimated prior probability to use the conformist strategy for a participant with a conformity score of zero. The prior probability that a participant with a conformity score of zero uses the conformist strategy is $exp(0.83)/(1 + exp(0.83)) = 0.69$.

The coefficient of the score reveals that the prior probability to use the conformist strategy increases with the score of the participant in the conformity questionnaire. To give an example, a participant who scores one standard deviation higher than average in the conformity questionnaire has a prior probability of $exp(0.83 + 0.87)/(1 + exp(0.83 + 0.87)) = 0.85$ to use the conformist strategy. The estimated prior probabilities are a function of the conformity score of each participant. The prior probabilities are returned by the estimation function as the object `model.DFS2020$prior.assignment`.

The function `stratEst.test()` can be used to show that the estimated coefficients significantly differ from zero.

```
R> test.coefficients <- stratEst.test(model.DFS2020, par = "coefficients")
R> print(test.coefficients)
```

```
                    estimate   diff std.error t.value  df p.value
coefficients.par.1    0.8274 0.8274    0.3065  2.6997 103  0.0081
coefficients.par.2    0.8697 0.8697    0.3184  2.7319 103  0.0074
```

The function `stratEst.check()` can be used to assess the global and local model fit based on the Pearson χ^2 test statistic.

```
R> check.DFS2020 <- stratEst.check(model.DFS2020, chi.tests = TRUE,
+                                  bs.samples = 100)
R> print(check.DFS2020$chi.global)

                    chi^2        min     mean      max p.value
model.DFS2020 0.08554165 0.07108578 2.623929 8.065177    0.99

R> print(check.DFS2020$chi.local)

                   chi^2      min      mean       max p.value
anticonformist  52.29308 17.60894  47.81437  79.03572    0.38
conformist     117.70654 58.92359 109.50872 155.70476    0.31
```

The distribution of each test statistic is approximated by 100 bootstrap samples. The p value of the global test indicates the probability of the data, under the assumption that the estimated model is the true model. The p value of the local tests indicate the probability that the choices of all participants which are assigned to the same strategy are generated by this strategy. Both tests support the null hypothesis that the estimated model is the true data generating model.

4. Conclusion

The R package **stratEst** provides an easy-to-use framework for fitting finite mixture models of choice strategies to discrete choice data. The estimation function of the package fits a finite mixture model of user defined choice strategies and returns various quantities of interest, like the maximum likelihood estimates and the standard errors of the model parameters. The estimation function can also be used to fit strategy estimation models with covariates to study the individual determinants of strategy use. The package provides functions which facilitate strategy programming, data processing and simulation, model selection and checking, and parameter testing.

The package is still under development. One restriction that will be relaxed in future versions of the package is that all strategies of a model need to react to the same input. In many cases, it is possible to represent all strategies of a model as automata that jointly fulfill this restriction. However, the automaton representation of some strategies might be unnecessarily complex in this case, which makes the programming of the strategies more difficult than it should be.

Another planned extension is the implementation of additional model checks for the strategy estimation model with covariates. Currently, the checks that can be performed for the model with covariates focus on the measurement model and test the global model fit and the assumption of conditional independence. In addition to conditional independence, the structural part of the model with covariates assumes non-differential measurement. Non-differential measurement suggests that the choices of all individuals that use the same strategy are not associated with the covariates of these individuals. A straightforward test for this assumption would be to regress the choices of all individuals that use the same strategy on the covariates of these individuals and check for significant coefficients. The fact that the true assignments of individuals to strategies are unknown, creates a problem for this test. One solution is to classify individuals according to the posterior probability assignments of individuals to strategies returned by the fitted model (Bandeen-Roche *et al.* 1997).

5. Function documentation

5.1. stratEst.strategy()

The strategy generation function of the package. The syntax of the function call is:

```
R> stratEst.strategy(choices, inputs = NULL, prob.choices = NULL,
+                    tr.inputs = NULL, trembles = NULL, num.states = 1)
```

Inputs

`choices:`	a character vector. The levels of the factor `choice` in the data.
`inputs:`	a character vector. The levels of the factor `input` in the data.
`prob.choices:`	a numeric vector. The choice probabilities of the strategy in row wise order.
`tr.inputs:`	a vector of integers. The deterministic state transitions of the strategy in row wise order.
`trembles:`	a numeric vector. The tremble probabilities of the strategy.
`num.states:`	an integer. The number states of the strategy.

Outputs

A `stratEst.strategy` object. A data frame with the following variables:

`prob.x`	the probability of choice `x`.
`tremble:`	the probability to observe a tremble.
`tr(x):`	the deterministic state transitions of the strategy for input `x`.

5.2. stratEst.data()

The data generation function of the package. The syntax of the function call is:

```
R> stratEst.data(data, choice = "choice", id = "id", input, input.lag = 0,
+                input.sep = "", game = "game", period = "period",
+                add = NULL, drop = NULL)
```

Input

`data:`	a `data.frame` in the long format.
`choice:`	a character string. The variable in `data` which contains the discrete choices. Default is `"choice"`.
`input:`	a character string. The names of the input generating variables in `data`. At least one input generating variable has to be specified. Default is `c("input")`.
`input.lag:`	a numeric vector. The time lag in periods of the input generating variables. The vector must have as many elements as variables specified in the object `input`. Default is zero.
`input.sep:`	a character string. Separates the input generating variables. Default is `""`.
`id:`	a character string. The name of the variable in `data` that identifies observations of the same individual. Default is `"id"`.
`game:`	a character string. The name of the variable in `data` that identifies observations of the same game. Default is `"game"`.
`period:`	a character string. The name of the variable in `data` that identifies the periods of a game. Default is `"period"`.
`add:`	a character vector. The names of variables in the global environment that should be added to the `stratEst.data` object. Default is `NULL`.
`drop:`	a character vector. The names of variables in `data` that should be dropped. Default is `NULL`.

Output

A `stratEst.data` object. A data frame in the long format with the following variables:

`id:`	the variable that identifies observations of the same individual.
`game:`	the variable that identifies observations of the same game.
`period:`	the period of the game.
`choice:`	the discrete choices.
`input:`	the inputs.

5.3. stratEst.model()

The estimation function of the package. The syntax of the function call is:

```
R> stratEst.model(data, strategies, shares = NULL, coefficients = NULL,
+                 covariates = NULL, sample.id = NULL, response = "mixed",
+                 sample.specific = c("shares","probs","trembles"),
+                 r.probs = "no", r.trembles = "global", select = NULL,
+                 outer.tol = 1e-10, outer.max = 1000, inner.runs = 10,
+                 inner.tol = 1e-5, inner.max = 10, lcr.runs = 100,
+                 lcr.tol = 1e-10, lcr.max = 1000 , bs.samples = 1000,
+                 quantiles = c(0.025,0.5,0.975), stepsize = 1,
+                 penalty = FALSE, verbose = TRUE)
```

Input

`data:`	a `stratEst.data` object or `data.frame`.
`strategies:`	a list of strategies. Each element if the list must be an object of class `stratEst.strategy`.
`shares:`	a numeric vector of strategy shares. The order of the elements corresponds to the order in `strategies`. Elements which are `NA` are estimated from the data. Use a list of numeric vectors if data has more than one sample and shares are sample specific.
`coefficients:`	a matrix of latent class regression coefficients.
`covariates:`	a character vector with the names of the covariates of the latent class regression model in the data. The covariates cannot have missing values.
`sample.id:`	a character string indicating the name of the variable which identifies the samples in data. Individual observations must be nested in samples.
`response:`	a character string which is either `"pure"` or `"mixed"`. If `"pure"` the estimated choice probabilities are either zero or one. If `"mixed"` the estimated choice probabilities are mixed parameters. The default is `"mixed"`.
`sample.specific:`	a character vector. Defines the model parameters that are sample specific. Can contain the character strings `"shares"` (`"probs"`, `"trembles"`. If the vector contains `"shares"` (`"probs"`, `"trembles"`), the estimation function estimates a set of shares (choice probabilities, trembles) for each sample in the data.
`r.probs:`	a character string. Options are `"no"`, `"strategies"`, `"states"` or `"global"`. Option `"no"` yields one vector of choice probabilities per strategy and state. Option `"strategies"` yields one vector of choice probabilities per strategy. Option `"states"` yields one vector of choice probabilities per state. Option `"global"` yields a single vector of choice probabilities. Default is `"no"`.

`r.trembles:`	a character string. Options are `"no"`, `"strategies"`, `"states"` or `"global"`. Option `"no"` yields one tremble probability per strategy and state. Option `"strategies"` yields one tremble probability per strategy. Option `"states"` yields one tremble probability per state. Option `"global"` yields a single tremble probability. Default is `"no"`.
`select:`	a character vector. Indicates the classes of model parameters that are selected. Can contain the strings `"strategies"`, `"probs"`, and `"trembles"`. If the vector contains`"strategies"` (`"probs"`, `"trembles"`), the number of strategies (choice probabilities, trembles) is selected based on the selection criterion specified in `"crit"`. The selection can be restricted with the arguments `r.probs` and `r.trembles`. Default is `NULL`.
`min.strategies:`	an integer. The minimum number of strategies in case of strategy selection. The strategy selection procedure stops if the minimum is reached.
`crit:`	a character string. Defines the information criterion used for model selection. Options are `"bic"` (Bayesian information criterion), `"aic"` (Akaike information criterion) or `"icl"` (Integrated Classification Likelihood). Default is `"bic"`.
`se:`	a string. Defines how standard errors are obtained. Options are `"analytic"` or `"bootstrap"`. Default is `"analytic"`.
`outer.runs:`	an integer. The number of outer runs of the solver. Default is 1.
`outer.tol:`	a number close to zero. The tolerance of the stopping condition of the outer runs. The iterative algorithm stops if the relative decrease of the log likelihood is smaller than this number. Default is 1e-10.
`outer.max:`	an integer. The maximum number of iterations of the outer runs of the solver. The iterative algorithm stops after `"outer.max"` iterations if it does not converge. Default is 1000.
`inner.runs:`	an integer. The number of inner runs of the solver. Default is 10.
`inner.tol:`	a number close to zero. The tolerance of the stopping condition of the inner runs. The iterative algorithm stops if the relative decrease of the log likelihood is smaller than this number. Default is 1e-5.
`inner.max:`	an integer. The maximum number of iterations of the outer runs of the solver. The iterative algorithm stops after `"inner.max"` iterations if it does not converge. Default is 10.

`lcr.runs:`	an integer. The number of latent class regression runs of the solver. Default is 100.
`lcr.tol:`	a number close to zero. The tolerance of the stopping condition of the latent class regression runs. The iterative algorithm stops if the relative decrease of the log likelihood is smaller than this number. Default is 1e-10.
`lcr.max:`	an integer. The maximum number of iterations of the latent class regression runs of the solver. The iterative algorithm stops after `"lcr.max"` iterations if it does not converge. Default is 1000.
`bs.samples:`	an integer. The number of bootstrap samples.
`quantiles:`	a numeric vector. The quantiles of the sampling distribution of the estimated parameters. Depending on the option of `se`, the quantiles are either estimated based on a t-distribution with `res.degrees` of freedom and the analytic standard errors or based the bootstrap.
`step.size:`	a number between zero and one. The step size of the Fisher scoring step which updates the coefficients. Values smaller than one slow down the convergence of the algorithm and prevent overshooting. Default is one.
`penalty:`	a logical. If `TRUE` the Firth penalty is used to estimate the coefficients of the latent class regression model. Default is `FALSE`.
`verbose:`	a logical. If `TRUE` information about the estimation process are printed to the console. Default is `FALSE`.

Output

An object of class `stratEst`. A list with the following elements.

`strategies:`	the fitted strategies.
`shares:`	the strategy shares.
`probs:`	the choice probabilities of the strategies.
`trembles:`	the tremble probabilities of the strategies.
`gammas:`	the gamma parameters of the strategies.
`coefficients:`	the coefficients of the covariates.
`shares.par:`	the estimated strategy share parameters.
`probs.par:`	the estimated choice probability parameters.

`trembles.par:`	the estimated tremble parameters.
`gammas.par:`	the estimated gamma parameters.
`coefficients.par:`	the estimated coefficient parameters of the covariates.
`shares.indices:`	the parameter indices of the strategy shares.
`probs.indices:`	the parameter indices of the choice probabilities.
`trembles.indices:`	the parameter indices of the tremble probabilities.
`coefficients.indices:`	the parameter indices of the coefficients.
`loglike:`	the log likelihood of the model.
`num.ids:`	the number of individuals.
`num.obs:`	the number of observations.
`num.par:`	the total number of model parameters.
`free.par:`	the number of free model parameters.
`res.degrees:`	the residual degrees of freedom.
`aic:`	the Akaike information criterion.
`bic:`	the Bayesian information criterion.
`icl:`	The integrated classification likelihood.
`crit.val:`	the value of the selection criterion defined by the argument `crit`.
`eval:`	the total number of iterations of the solver.
`tol.val:`	the relative decrease of the log likelihood in the last iteration of the algorithm.
`convergence:`	the maximum of the absolute scores of the estimated model parameters.
`entropy.model:`	the entropy of the model.
`entropy.assignments:`	the entropy of the posterior probability assignments of individuals to strategies.
`chi.global:`	the chi square statistic for global model fit.
`chi.local:`	the chi square statistics for local model fit.
`state.obs:`	the weighted observations for each strategy state.
`post.assignments:`	the posterior probability assignments of individuals to strategies.

`prior.assignments:`	the prior probability of each individual to use a strategy as predicted by the individual covariates.
`shares.se:`	the standard errors of the estimated share parameters.
`probs.se:`	the standard errors of the estimated choice probability parameters.
`trembles.se:`	the standard errors of the estimated tremble probability parameters.
`gammas.se:`	the standard errors of the estimated gamma parameters.
`coefficients.se:`	the standard errors of the estimated coefficients.
`shares.quantiles:`	the quantiles of the estimated population shares.
`probs.quantiles:`	the quantiles of the estimated choice probabilities.
`trembles.quantiles:`	the quantiles of the estimated trembles.
`coefficients.quantiles:`	the quantiles of the estimated coefficients.
`shares.score:`	the scores of the estimated share parameters.
`probs.score:`	the score of the estimated choice probabilities.
`trembles.score:`	the score of the estimated tremble probabilities.
`coefficients.score:`	the score of the estimated coefficients.
`shares.fisher:`	the Fisher information matrix of the estimated shares.
`probs.fisher:`	the Fisher information matrix of the estimated choice probabilities.
`trembles.fisher:`	the Fisher information matrix of the estimated trembles.
`coefficients.fisher:`	the Fisher information matrix of the estimated coefficients.
`fit.args:`	the input objects of the function call.

5.4. stratEst.simulate()

The simulation function of the package. The syntax of the function call is:

```
R> stratEst.simulate( data = NULL, strategies, shares = NULL,
+                     coefficients = NULL, covariates = NULL,
+                     num.ids = 100, num.games = 5, num.periods = NULL,
+                     fixed.assignment = FALSE, input.na = FALSE)
```

Input

`data:`	a `stratEst.data` object. Alternatively, the arguments `num.ids`, `num.games`, and `num.periods` can be used if no data is available.
`strategies:`	a list of strategies. Each element if the list must be an object of class `stratEst.strategy`.
`shares:`	a numeric vector of strategy shares. The order of the elements corresponds to the order in `strategies`. `NA` values are not allowed. Use a list of numeric vectors if data has more than one sample and shares are sample specific.
`coefficients:`	a matrix of regression coefficients. Column names correspond to the names of the strategies, row names to the names of the covariates.
`covariate.mat:`	a matrix with the covariates in columns. The column names of the matrix indicate the names of the covariates. The matrix must have as many rows as there are individuals.
`num.ids:`	an integer. The number of individuals. Default is 100.
`num.games:`	an integer. The number of games. Default is 5.
`num.periods:`	a vector of integers with as many elements `num.games`. The elements specify the number of periods in each game. Default (`NULL`) means 5 periods in each game.
`fixed.assignment:`	a logical value. If `FALSE` individuals use potentially different strategies in each game. If `TRUE`, individuals use the same strategy in each game. Default is `FALSE`.
`input.na:`	a logical value. If `FALSE` an input value is randomly selected for the first period. Default is `FALSE`.
`sample.id:`	a character string indicating the name of the variable which identifies the samples in data. Individual observations must be nested in samples. Default is `NULL`.

Output

A `stratEst.data` object. A data frame in the long format with the following variables:

`id:`	the variable that identifies observations of the same individual.
`game:`	the variable that identifies observations of the same game.
`period:`	the period of the game.
`choice:`	the discrete choices.
`input:`	the inputs.

`sample:`	the sample of the individual.
`strategy:`	the strategy of the individual.

5.5. stratEst.test()

The test function of the package. The syntax of the function call is:

```
R> stratEst.test(model, par = c("shares", "probs", "trembles",
+                                "coefficients"), values = 0,
+                alternative = "two.sided", digits = 4)
```

Input

`model:`	a fitted model of class `stratEst.model`.
`par:`	a character vector. The class of model parameters to be tested. The default is to test all classes of model parameters.
`values:`	a numeric vector. The values the parameter estimates are compared to. Default is zero.
`alternative:`	a character string. The alternative hypothesis. Options are `"two.sided"`, `"greater"` or `"less"`. Default is `"two.sided"`.
`digits:`	an integer. The number of digits of the result.

Output

A `data.frame` with one row for each tested parameter and 6 variables:

`estimate:`	the parameter estimate.
`diff:`	the difference between the estimated parameter and the numeric value (if supplied).
`std.error:`	the standard error of the estimated parameter.
`t.value:`	the t statistic.
`res.degrees:`	the residual degrees of freedom of the model.
`p.value:`	the p value of the t statistic.

5.6. stratEst.check()

The function for model checking of the package. The syntax of the function call is:

```
R> stratEst.check( model, chi.tests = F, bs.samples = 100, verbose = FALSE )
```

Input

`model:`	a fitted model of class `stratEst.model`.
`chi.tests:`	a logical. If `TRUE` chi square tests of global and local model fit are performed. Default is `FALSE`.
`bs.samples:`	an integer. The number of parametric bootstrap samples for the chi square tests. Default is 100.
`verbose:`	a logical, if `TRUE` messages of the checking process are printed to the console. Default is `FALSE`.

Output

A list of check results with the following elements:

`fit:`	a matrix. Contains the log likelihood, the number of free model parameters, and the value of the three information criteria.
`chi.global:`	a matrix. The results of the chi square test for global model fit.
`chi.local:`	a matrix. The results of the chi square test for local model fit.

Computational details

The results in this paper were obtained using R 4.1.1 with the **stratEst** 1.0.1 package. R itself and all packages used are available from the Comprehensive R Archive Network (CRAN) at `https://CRAN.R-project.org/`.

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