The mathematics of Stute (1997) test

Let Y and D be two random variables. Let $m(D) = E[Y|D]$. The null hypothesis of the test is that $m(D) = \alpha_0 + \alpha_1 D$ for two real numbers α_0 and α_1. This means that, under the null, $m(.)$ is linear in D. This hypothesis can be tested in a sample with N i.i.d. realizations of (Y, D) using the following test statistic from Stute (1997):

$$S = \frac{1}{N^2} \sum_{i=1}^{N} \left(\sum_{j=1}^{i} \varepsilon_{(j)} \right)^2$$

where $\varepsilon_{(j)}$ is the residual from a linear regression of Y on D and a constant of the j-th observation after sorting by D. In other words, S is obtained by sorting the data from the smallest to the largest value of D and summing the squares of the total cumulative sums of the linear regression residuals.

Stute et al. (1998) show that, under the null, S is finite. Conversely, under the alternative, at least one of the inner sums tends to infinity, hence S diverges. Inference is performed using wild bootstrap. Specifically, S is re-computed replacing Y with Y^*, i.e. the predicted value of Y from the linear regression of Y on D and a constant, plus the residuals multiplied by a two-point random variable, denoted as $V_{(j)}$, such that:

$$P\left(V_{(j)} = \frac{1+\sqrt{5}}{2}\right) = \frac{\sqrt{5}-1}{2}, P\left(V_{(j)} = \frac{1-\sqrt{5}}{2}\right) = \frac{3-\sqrt{5}}{2}.$$

Denote with S^*_b the S^* statistic computed at the b-th bootstrap replication. The p-value from B bootstrap replications is computed as

$$\frac{1}{B} \sum_{b=1}^{B} 1\{S < S^*_b\}$$

Intuitively, under the alternative, the p-value should be zero, due to the fact that S diverges.

This test also works with panel data. In that case, the S statistic is computed for each value of the time variable. Moreover, $V_{(j)}$ remains constant at the group level across the computation of the period-specific test statistics. Hence, the residual of group g from a linear regression of $Y_{g,t}$ on $D_{g,t}$ and a constant are multiplied by V_g, regardless of t. Lastly, the individual test results can be summed into a joint test statistic. In this case, inference is performed using the distribution of the sum of the bootstrap statistics. Denote with S_ℓ the period-ℓ test statistic and with $S^*_{\ell,b}$ its b-th bootstrap estimate. In a dataset with L periods, the p-value of the joint test is computed as follows:

$$\frac{1}{B}\sum_{b=1}^{B} 1\left\{\sum_{\ell=1}^{L} S_\ell < \sum_{\ell=1}^{L} S^*_{\ell,b}\right\}.$$

References

- Stute, W. (1997). Nonparametric model checks for regression. The Annals of Statistics.
- Stute, W., W. G. Manteiga, and M. P. Quindimil (1998). Bootstrap approximations in model checks for regression. Journal of the American Statistical Association.